Classification of topological insulators and superconductors and D-branes

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phases and phase transitions in condensed matter systems

classical phases

Ginzberg-Laudau theory

Nambu-Goldstone modes

quantum phases

gapless phases

- Fermi liquid
- non Fermi liquid

gapped phases

- insulators
- topological insulators
- topological superconductors
- topological phases

quantum critical points

- relativistic conformal quantum critical point
- quantum Lifshitz critical point

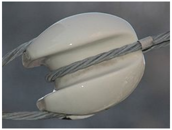
table of contents

- topological phases in condensed matter:
  distinguishing states in terms of wavefnctions

- table of topological insulators and superdoncutors

- connection to D-branes

insulator: material which resists the flow of electric current.

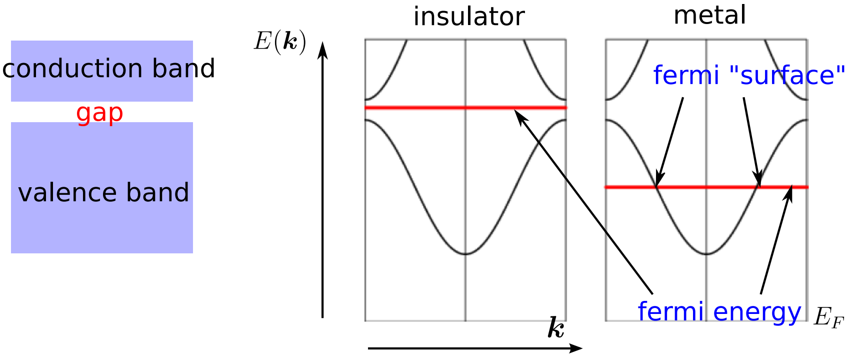


(gapped) superconductor: "band insulator" for fermionic quasiparticles

band theory and band insulator

Bloch (1928), Wilson (1931)

single partcle energy spectrum of an electron in solids
= "band structure"



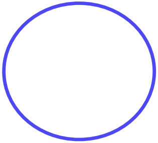
crystal momentum in
1st Brillouin Zone, BZ

Role of electric wavefucntions in insulators ???

topological insulator: preview

distinction of insulators by their wavefunctions (or: entanglement)

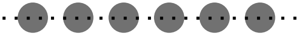
--> "topological insulators"



$\neq$

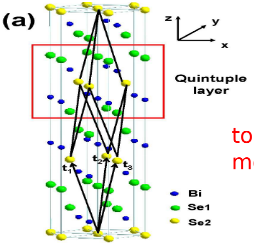
=

$|\Psi\rangle$ $|\Psi_1\rangle$ $|\Psi_2\rangle$



boring insulator
less entangled

$\neq$



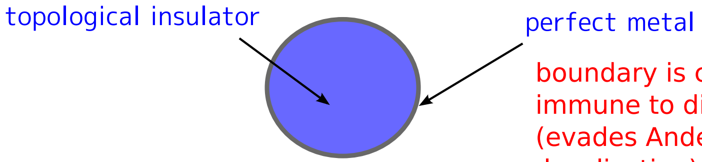
topological insulator
more entangled

bulk-boundary correspondence

physical consequence of entangled wavefunctions

boundary of ordinary insulator = insulator

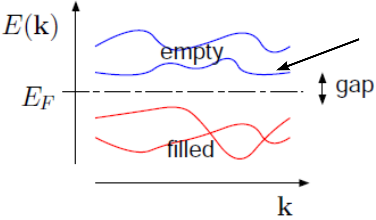
boundary of topological insulator = perfect metal



boundary is completely immune to disorder (evades Anderson localization).

A consequence: ordinary and topological insulators cannot be connected adiabatically.

Bloch wavefunction bundle



$|u_a(k)\rangle$ "Bloch wavefunction"

a map from BZ to the space of wavefunctions (projectors)

$$P(k) = \sum_a u_a(k) u_a^\dagger(k) \in U(n+m)/U(n)\times U(m)$$

n,m: number of bands

possible obstruction for constructing a smooth wavefunction over BZ

$$\mathrm{Ch}_{n+1}[\mathcal{F}] = \int_{\mathrm{BZ}^{d=2n+2}} \mathrm{ch}_{n+1}(\mathcal{F}) = \int_{\mathrm{BZ}^{d=2n+2}} \frac{1}{(n+1)!}\,\mathrm{tr}\left(\frac{\mathrm{i}\mathcal{F}}{2\pi}\right)^{n+1} \in \mathbb{Z}.$$

$$\mathcal{A}^{\hat{a}\hat{b}}(k) = A^{\hat{a}\hat{b}}_\mu(k)\mathrm{d}k_\mu = \langle u^-_{\hat{a}}(k)|\mathrm{d}u^-_{\hat{b}}(k)\rangle$$

Berry gauge field (k-space gauge field)

E.g. quantum Hall effect

$$\sigma_{xy} = \frac{i}{2\pi}\int_{\mathrm{BZ}^2} \mathrm{tr}\,\mathcal{F}(k)$$

$$\pi_2\left[U(m+n)/U(m)\times U(n)\right] = \mathbb{Z}$$

discrete symmetries

E.g.: $\{\mathcal{H}(k), \Gamma\} = 0, \quad \Gamma^2 = 1$

$$\mathcal{H}(k) = \begin{pmatrix} 0 & D(k) \\ D^\dagger(k) & 0 \end{pmatrix}$$

topological invariant: $\pi_{d=\text{odd}}\left[U(m)\right] = \mathbb{Z}$

$$\nu_{2n+1}[q] := \int_{\text{BZ}^{d=2n+1}} \omega_{2n+1}[q]$$

$$\omega_{2n+1}[q] := \frac{(-1)^n n!}{(2n+1)!}\left(\frac{\mathrm{i}}{2\pi}\right)^{n+1} \operatorname{tr}\left[(q^{-1}\mathrm{d}q)^{2n+1}\right]$$

$$\begin{cases} DD^\dagger u_a = \lambda^2 u_a \\ D^\dagger D v_a = \lambda^2 v_a \end{cases}$$

$$q(k) = \sum_a u_a(k) v_a^\dagger(k) \in U(m)$$

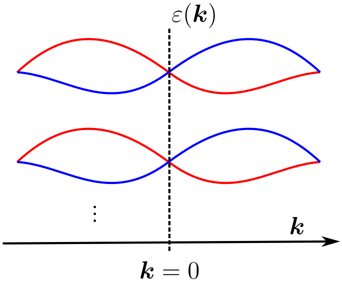
topological phases in odd space dimensions

discrete symmetry relating k and -k:

E.g.: time-reversal

$$i\sigma_y \mathcal{H}^*(-k)(-i\sigma_y) = \mathcal{H}(k)$$

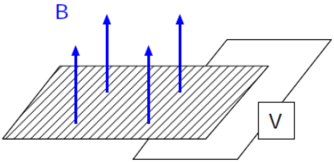
topological phases with Z2 classification



integer quantum Hall effect (IQHE)

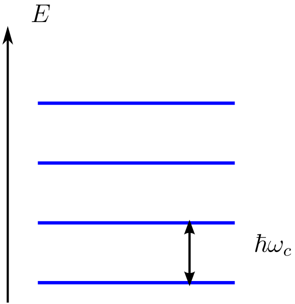
K.v.Klitzing, G. Dorda, M. Pepper (1980)

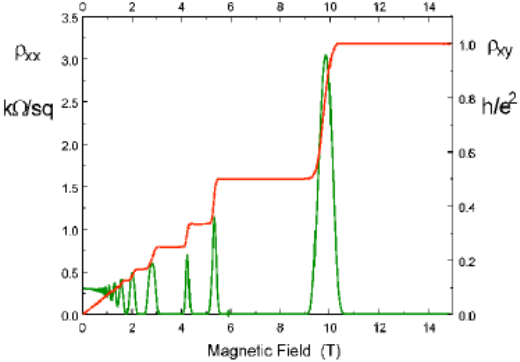
in d=2 spatial dimensions, with strong T breaking by B



$$\sigma_{xx} = 0 \qquad \rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} = 0$$

$$\boxed{\sigma_{xy} = \frac{e^2}{h} \times (\text{integer})}$$

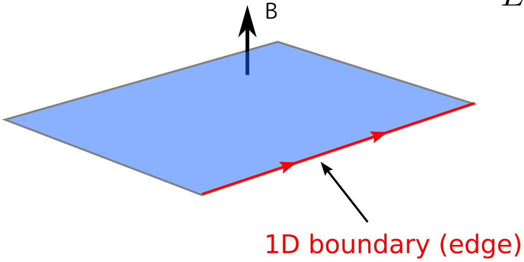


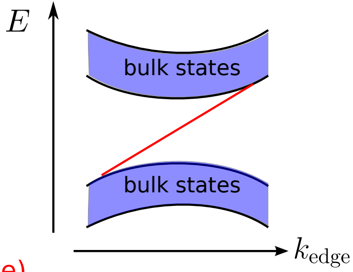


integer quantum Hall edge states

Secret behind quanitization: edge states

There is a gapless chiral edge mode along the samlple boundary.



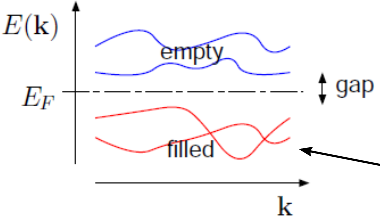


Number of edge modes $= \dfrac{-\sigma_{xy}}{e^2/h} = C$

Robust against disorder (chiral fermions cannot be backscatterd).

IQHE as a topological insulator

- "bulk" point of view



"Bloch wavefunction"

$|u_a(k)\rangle$

Hall conductance = topological invariant ! "Chern number"

$$\sigma_{xy} = \frac{e^2}{h} \sum_{a \in \text{bands}}^{\text{filled}} \int_{\text{BZ}} \frac{d^2k}{2\pi i} \left[\left\langle \frac{\partial u_a(k)}{\partial k_y} \middle| \frac{\partial u_a(k)}{\partial k_x} \right\rangle - \left\langle \frac{\partial u_a(k)}{\partial k_x} \middle| \frac{\partial u_a(k)}{\partial k_y} \right\rangle \right]$$

Thouless-Kohmoto-Nightingale-den Nijs (TKNN) (1982)

Bloch wavefunctions define a map from BZ to the space of wavefunctions (projection operators).

$$\pi_2 \left[U(m+n)/U(m) \times U(n) \right] = \mathbb{Z}$$

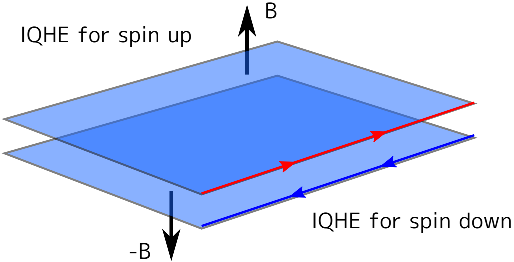
quantum spin Hall effect (QSHE)

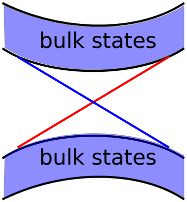
in d=2 spatial dimensions, with good T

TRS

$$(i\sigma_y)\mathcal{H}^*(-i\sigma_y) = \mathcal{H}$$

- time-reversal invariant band insulator
- gapless Kramers pair of edge modes
- strong spin-orbit interaction





quantum spin Hall effect (QSHE)

in d=2 spatial dimensions, with good T

quantum spin Hall insulator is characterized by a binary ($\mathbb{Z}_2$) topological quantity.

Kane-Mele (05)

- odd number of Kramers pairs at edge --> stable

  even number of Kramers pairs at edge --> unstable

1+1=0 3 = 1

experimental realization:
HgTe quantum well

Bernevig-Hughes-Zhang (2006)
M. Koenig et al. Science (2007)

quantum spin Hall effect (QSHE)

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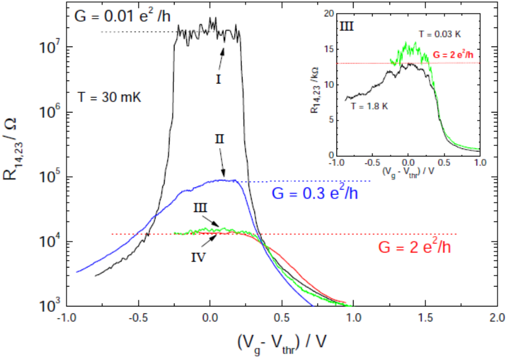


Z2 topological insulator in d=3 spatial dimensions

Fu-Kane-Mele, Moore-Balents, Roy (06)

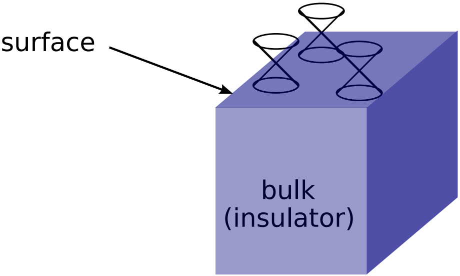
d=3 dimensions

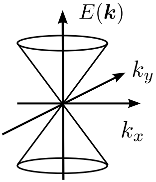
time-reversal invariant $i\sigma_y \mathcal{H}^* (-i\sigma_y) = \mathcal{H}$

characterized by a Z2 quantity $\nu_0 = 0 \,\mathrm{or}\, 1$

trivial (0) non-trivial (1)

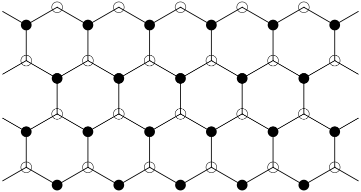
when $\nu_0 = 1$ surface states = odd number of Dirac fermions

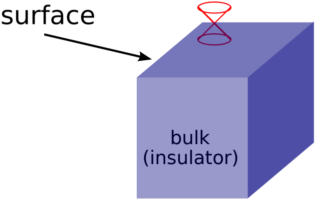




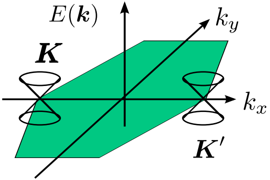
$$\mathcal{H} = v_F \left(\sigma_x p_x + \sigma_y p_y \right)$$

surface of top. insulator = "1/4 of graphene" !



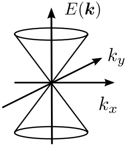


two valleys
(and spin)



$$\mathcal{H}_0 = \sum_{s=\uparrow,\downarrow} \psi_s^\dagger \begin{pmatrix} \tau_x k_x + \tau_y k_y & 0 \\ 0 & -\tau_x k_x - \tau_y k_y \end{pmatrix} \psi_s$$

Theorem (by Nielsen-Ninomiya):
For any 2D lattice with TRS
# of Dirac cones must be even.



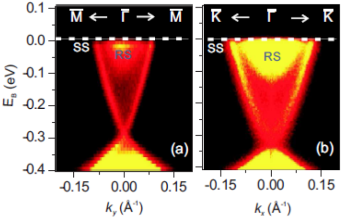
$$\mathcal{H} = v_F \left(\sigma_x p_x + \sigma_y p_y\right)$$

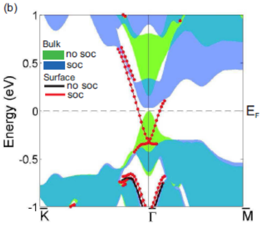
condensed matter realization of
domain-wall fermion

ARPES experiments on Z2 topological insulators

BiSb D. Hsieh et al. Nature (08) 5 Dirac cones

BiSe Y. Xia et al. Nature Phys. (2009)





BiTe Y. L. Chen et al. Science (2009)

Summary so far...

Imposing time-resersal symmetry --> new topological insulators

- IQHE (1980) — K.v.Klitzing, G. Dorda, M. Pepper (1980) — SiMOS,GaAs
- QSHE (2007) — B.Bernevig, T.Hughes, S.C.Zhang (06); M. Konig et al. (07); Kane and Mele et al. (05-06) — HgTe
- 3D Z2 topological insulator (2008) — BiTe, BiSe, BiSb
  Moore-Balents (07) Fu-Kane-Mele (07) Roy (07)
  D. Hsieh et al (08)
- chiral p-wave SC (topological superconductor) — SrRuO; nu=5/2 FQHE
  Y. Maeno et al (94)
  R.L.Willett et al (87) G. Moore and N. Read et al (91)

more topological insulators/superconductors ?

never ending story or happy ending ?

discrete symmetries

two types of anti-unitary symmetries

Time-Reversal Symmetry (TRS)

$$\mathcal{T}\mathcal{H}^*\mathcal{T}^{-1} = \mathcal{H}$$

TRS = { 0 no TRS; +1 TRS with $\mathcal{T}^T = +\mathcal{T}$ (integer spin particle); -1 TRS with $\mathcal{T}^T = -\mathcal{T}$ (half-odd integer spin particle) }

Particle-Hole Symmetry (PHS)

$$C\mathcal{H}^T C^{-1} = -\mathcal{H}$$

PHS = { 0 no PHS; +1 PHS with $C^T = +C$; -1 PHS with $C^T = -C$ }

PHS + TRS = chiral symmetry

$$\left.\begin{aligned} T\mathcal{H}^*T^{-1} &= \mathcal{H} \\ C\mathcal{H}^*C^{-1} &= -\mathcal{H} \end{aligned}\right\} \longrightarrow TC\mathcal{H}(TC)^{-1} = -\mathcal{H}$$

particle-hole symmetry; examples

generic SC: $H = \frac{1}{2}\int \Psi^\dagger \mathcal{H} \Psi$ $\qquad \mathcal{H} = \begin{pmatrix} \xi & \Delta \\ -\Delta^* & -\xi^T \end{pmatrix}$

Nambu spinor: $\Psi^\dagger = \begin{pmatrix} \psi^\dagger_\uparrow, & \psi^\dagger_\downarrow, & \psi_\uparrow, & \psi_\downarrow \end{pmatrix}$

particle-hole symmetry (PHS): $\Psi = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi^\dagger \right]^T$ PHS = +1

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{H}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\mathcal{H}$$

Sz-conserving SC: $H = \int \Psi^\dagger \mathcal{H} \Psi$ $\quad \mathcal{H} = \begin{pmatrix} \xi_\uparrow & \Delta \\ \Delta^\dagger & -\xi^T_\downarrow \end{pmatrix}$

$$\Psi^\dagger = \begin{pmatrix} \psi^\dagger_\uparrow, & \psi_\downarrow \end{pmatrix}$$

random matrix ensembles - "ten-fold way"

	sym. class	TRS	PHS	SLS	description
Wigner-Dyson	A	0	0	0	unitary
	AI	+1	0	0	orthogonal
	AII	-1	0	0	symplectic (spin-orbit)
chiral	AIII	0	0	1	chiral unitary
	BDI	+1	+1	1	chiral orthogonal
	CII	-1	-1	1	chiral symplectic
BdG	D	0	+1	0	singlet/triplet SC
	C	0	-1	0	singlet SC
	DIII	-1	+1	1	singlet/triplet SC with T
	CI	+1	-1	1	singlet SC with TRS

- Wigner-Dyson (1951 -1963) : "three-fold way" — complex nuclei
- Verbaarschot (1992 -1993) — chiral phase transition in QCD
- Altland-Zirnbauer (1997) : "ten-fold way" — mesoscopic SC systems

claim: this is the exhaustive classification of discrete symmetries

BdG Hamiltonians realize 6 out of 10 symmetry classes.

classification of topological insulators and superconductors

AZ\d	1	2	3
A	0	$\mathbb{Z}$	0
AIII	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	0	0	0
BDI	$\mathbb{Z}$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	$\mathbb{Z}$	0
CI	0	0	$\mathbb{Z}$

$\mathbb{Z}$ integer classification

$\mathbb{Z}_2$ binary classification

0 no top. ins./SC

classification of topological insulators and superconductors

spatial dimensions

presence/absence of topological state

AZ\d	1	2	3
A	0	$\mathbb{Z}$	0
AIII	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	0	0	0
BDI	$\mathbb{Z}$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	$\mathbb{Z}$	0
CI	0	0	$\mathbb{Z}$

symmetry classes of quadratic fermionic Hamiltonians (Altland-Zirnbauer)

$\mathbb{Z}$ integer classification

$\mathbb{Z}_2$ binary classification

0 no top. ins./SC

classification of topological insulators and superconductors

AZ\ d	1	2	3
A	0	$\mathbb{Z}$	0
AIII	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	0	0	0
BDI	$\mathbb{Z}$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	$\mathbb{Z}$	0
CI	0	0	$\mathbb{Z}$

IQHE

p+ip wave SC

polyacetylene

TMTSF

QSHE

Z2 topological insulator

d+id wave SC

classification of topological insulators and superconductors

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$

10 = 8 + 2

AZ\d	0	1	2	3	4	5	6	7	8	9
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

- periodicity 8 both in spatial dimension and symmetry class

- always 5 kinds of topological states for each dimension.

- Z followed by two Z2 ("dimensional reduction")

- d>3 can characterize adiabatic processes, rather than states themselves

- Schnyder, SR, Furusaki, Ludwig (for d=1,2,3, 2008)
- Kitaev (all d and periodicity "Periodic Table", 2009)
- Qi, Hughes, Zhang (cases with one symmetry, field theory description, 2008)
- SR and Takayanagi (construction by D-branes, 2010)

underlying strategies for classification

- discover a topological invariant

  - obtained 3D analogue of TKNN integer:

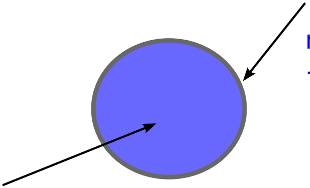
$$\nu = \int_{\mathrm{BZ}} \frac{d^3k}{24\pi^2} \epsilon^{\mu\nu\lambda} \mathrm{tr}\left[q^{-1}(\boldsymbol{k})\partial_\mu q(\boldsymbol{k}) q^{-1}(\boldsymbol{k})\partial_\nu q(\boldsymbol{k}) q^{-1}(\boldsymbol{k})\partial_\lambda q(\boldsymbol{k}) \right]$$

  - complication by $\boldsymbol{k} \equiv -\boldsymbol{k}$

- bulk-boundary correspondence

Anderson delocalization

non-linear sigma model on G/H
+ (discrete) topological term

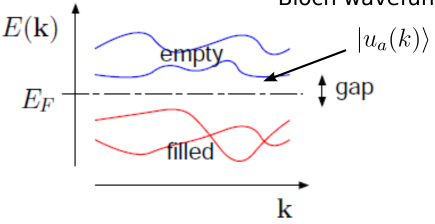


topological insulators/SC

why periodicity ? and why K-theory ?

Kitaev (2009)

"Bloch wavefunction"



a map from BZ to the space of wavefunctions (projectors).

e.g. $\pi_2\left[U(m+n)/U(m)\times U(n)\right] = \mathbb{Z}$

n,m: number of bands

By "adding" topologically trivial bands should not change the topological nature of the system

--> consider "stably" equivalent classes of insulators

$$E \sim\sim F \Leftrightarrow E \oplus I^k = F \oplus I^l$$

Can think of "difference" E-F of two Hamiltonians:

$$(E,F) \sim (E',F') \Leftrightarrow E' = E \oplus H, F' = F \oplus H$$

Bott periodicity = periodicity of topological insulators/SCs

some outcomes of classification

AZ\d	0	1	2	3	4	5	6	7	8	9
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

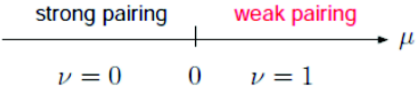
IQHE
p+ip wave SC
polyacetylene
3He B
TMTSF
Z2 topological insulator
QSHE
d+id wave SC
topological singlet SC

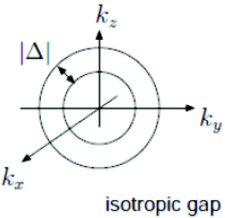
- 3He B is newly identified as a topological SC (superfluid) in d=3.
- topological singlet SC in d=3 is predicted.
- topological superconductors in non-centrosymmetric SCs.

3He B is a topological "superconductor" in class DIII

$$H = \frac{1}{2}\int d^3 r \Psi^\dagger \mathcal{H} \Psi \qquad \mathcal{H} = \begin{pmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi \end{pmatrix}$$

$$\xi_{\boldsymbol{k}} = \frac{\boldsymbol{k}^2}{2m} - \mu \qquad \Delta_{\boldsymbol{k}} = |\Delta| i\sigma_y \boldsymbol{k}\cdot\boldsymbol{\sigma}$$





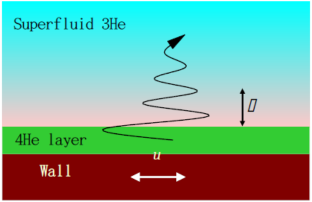
topologically protected
surface Majorana fermion

Schnyder, SR, Furusaki, Ludwig (2008)
Roy (2008)
Qi, Hughes, Raghu, Zhang (2008)

3d analogue of Moore-Read state

$$\Psi(\{\boldsymbol{r}_i\},\{\sigma_i\}) \sim \mathrm{Pf}\left(\frac{[(\boldsymbol{r}_i - \boldsymbol{r}_j)\cdot i\boldsymbol{\sigma}\sigma_y]_{\sigma_i\sigma_j}}{|\boldsymbol{r}_i - \boldsymbol{r}_i|^3}\right)$$

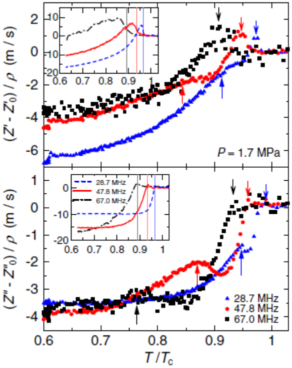
Majorana mode detected by surface acoustic impedance

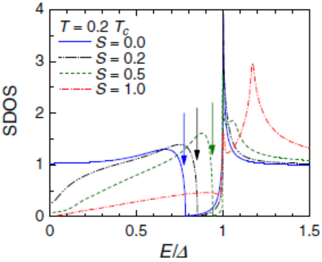


Y. Aoki et al. PRL (2005)

Y. Wada et al. PRB (2008)

S. Murakawa et al. PRL (2009)





Salomma and Volovik, 1980s

Y. Nagato et al. JLTP (2007)

M. Saitoh et al. PRB(R) (2006)

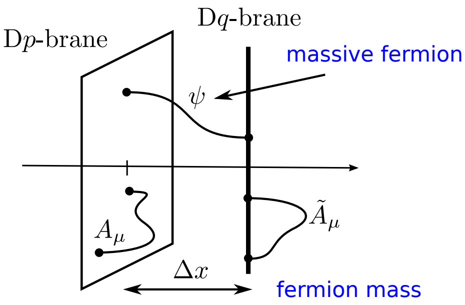
classification of D-branes

	D(−1)	D0	D1	D2	D3	D4	D5	D6	D7	D8	D9
type IIB	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$O9^-$ (type I)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$O9^+$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

TABLE III. Dp-brane charges from K-theory, classified by $K(\mathbb{S}^{9-p})$, $KO(\mathbb{S}^{9-p})$ and $KSp(\mathbb{S}^{9-p})$ [24]. A $\mathbb{Z}_2$ charged Dp-brane with p even or p odd represents a non-BPS Dp-brane or a bound state of a Dp and an anti-Dp brane, respectively [26].

Sen, Witten, Horava (98-99)

Dp-Dq system



E.g. IQHE (d=2): p=q=5

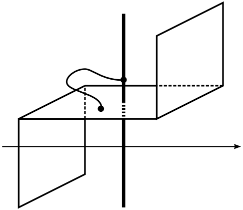
Dirac model of class A TI

$$\mathcal{L} = \bar{\psi}(\partial \cdot \gamma - A \cdot \gamma - \tilde{A} \cdot \gamma - m)\psi$$

$$\sigma_{xy} = \mathrm{sgn}\,(m)/2$$

$$\mathcal{L} = \frac{m}{8\pi|m|} \int A \wedge dA = \frac{m}{8\pi|m|} \int F \wedge F \wedge C^{(2)}_{RR}$$

edge state = intersection



N.B. A_μ : "external" gauge field
$\tilde{A}_\mu$: "internal" gauge field

Rey (2007), Davis et al (2008)
Bergman et al (2010), Fujita et al (2009)

one-to-one correspondence between TIs/TSCs and D-branes

Dp　Dq　spatial dimensions　symmetry class　SO orSp O-plane

	0	1	2	3	4	5	6	7	8	9	d	A
D5	×	×	×	×	×	×						
D3	×						×	×	×		0	$\mathbb{Z}$ (2 Mj)
D5	×	×	×				×	×	×		2	$\mathbb{Z}$ (2 Mj)
D7	×	×	×	×	×		×	×	×		4	$\mathbb{Z}$ (1 Di)
	0	1	2	3	4	5	6	7	8	9	d	AIII
D4	×		×	×	×	×						
D4	×		×				×	×	×		1	$\mathbb{Z}$ (2 Mj)
D6	×		×	×	×		×	×	×		3	$\mathbb{Z}$ (2 Mj)

	0	1	2	3	4	5	6	7	8	9	d	C ($O9^-$)	D ($O9^+$)
D5	×	×	×	×	×	×							
D3	×						×	×	×		0	0	$\mathbb{Z}_2$ (2 Mj)
D4	×	×					×	×	×		1	0	$\mathbb{Z}_2$ (1 Mj)
D5	×	×	×				×	×	×		2	$\mathbb{Z}$ (4 Mj)	$\mathbb{Z}$ (1 Mj)
D6	×	×	×	×			×	×	×		3	0	0
D7	×	×	×	×	×		×	×	×		4	$\mathbb{Z}_2$ (2 Di)	0
	0	1	2	3	4	5	6	7	8	9	d	CI ($O9^-$)	DIII ($O9^+$)
D5	×	×	×	×	×	×							
D2	×						×	×			0	0	0
D3	×	×					×	×			1	0	$\mathbb{Z}_2$ (2 Mj)
D4	×	×	×				×	×			2	0	$\mathbb{Z}_2$ (2 Mj)
D5	×	×	×	×			×	×			3	$\mathbb{Z}$ (4 Mj)	$\mathbb{Z}$ (1 Mj)
	0	1	2	3	4	5	6	7	8	9	d	AII ($O8^-$)	AI ($O8^+$)
D4	×	×	×	×	×								
D4	×						×	×	×	×	0	$\mathbb{Z}$ (4 Mj)	$\mathbb{Z}$ (1 Mj)
D5	×	×					×	×	×	×	1	0	0
D6	×	×	×				×	×	×	×	2	$\mathbb{Z}_2$ (4 Mj)	0
D7	×	×	×	×			×	×	×	×	3	$\mathbb{Z}_2$ (2 Mj)	0
D8	×	×	×	×	×		×	×	×	×	4	$\mathbb{Z}$ (1 Di)	$\mathbb{Z}$ (1 Di)
	0	1	2	3	4	5	6	7	8	9	d	CII ($O8^-$)	BDI ($O8^+$)
D4	×	×	×	×	×								
D3	×						×	×	×		0	0	$\mathbb{Z}_2$ (2 Mj)
D4	×	×					×	×	×		1	$\mathbb{Z}$ (4 Mj)	$\mathbb{Z}$ (1 Mj)
D5	×	×	×				×	×	×		2	0	0
D6	×	×	×	×			×	×	×		3	$\mathbb{Z}_2$ (4 Mj)	0
D7	×	×	×	×	×		×	×	×		4	$\mathbb{Z}_2$ (2 Di)	0

class A and AIII = Type IIA and IIB

real symmety classes = Type I

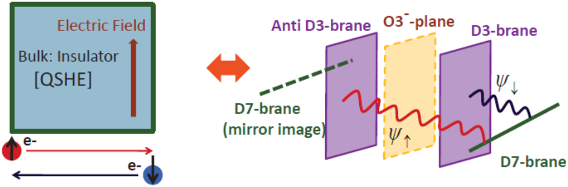
(i) PHS
= orientifold projection (SO or Sp)

(ii) SLS ("chiral symmetry")
= prarity (inversion)

(iii) TRS = orientifold x parity

"1st decendant" = non BPS D-brane

"2nd decendant" = brane-antibrane bound state   E.g. QSHE



primary series

1st decendant

2nd decendant

AZ\\d	0	1	2	3	4	5	6	7	8	9
AI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

internal and external gauge group

QHE, FQHE

Kitaev model (?)

gamma matrix Kitaev model (?)

G	class\ d	0	1	2	3	4	5	6	7
U	A	U	-	U	-	U	-	U	-
U	AIII	-	U	-	U	-	U	-	U
O	AI	O	-	-	-	Sp	-	U	O
O	BDI	O	O	-	-	-	Sp	-	U
O	D	U	O	O	-	-	-	Sp	-
O	DIII	-	U	O	O	-	-	-	Sp
Sp	AII	Sp	-	U	O	O	-	-	-
Sp	CII	-	Sp	-	U	O	O	-	-
Sp	C	-	-	Sp	-	U	O	O	-
Sp	CI	-	-	-	Sp	-	U	O	O

TABLE IV. External G (left-most column) and internal $\tilde{G}$ gauge groups for each spatial dimension d and symmetry class; U, O, Sp, represents U(1), O(1) $= \mathbb{Z}_2$, and Sp(1) $=$ SU(2), respectively.

c.f. projective construction of FQHE by X. G. Wen

$$\mathcal{L} = \bar{\psi}\big(\partial \cdot \gamma - A \cdot \gamma - \tilde{A} \cdot \gamma - m\big)\psi$$

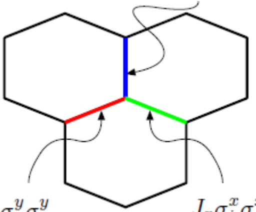
J. Maciejko et al C. Hoyos-Badajoz et al

B. Swingle et al A Karch et al

honeycomb lattice Kitaev model in 2 dimensions

Alexei Kitaev, Ann. Phys. (2005)

$$H = \sum_{\mu=1}^{3} J_\mu \sum_{\mu-\mathrm{links}} \sigma_i^\mu \sigma_j^\mu$$



- exactly solvable in terms of projective construction (emergent fermions)
- two phases: Abelian and non-Abelian phases
- supports Abelian and non-Abelian anyons as a quasi particle excitation

Kitaev model (purely bosonic model) = fermion + Z2 gauge field

introduce four Majorana fermions $\lambda^{1,2,3,4}$ $\lambda^{a\dagger} = \lambda^a$ $\lambda^{a2} = 1$

$$\sigma^{a=1,2,3} = i\lambda^{a=1,2,3}\lambda^4$$

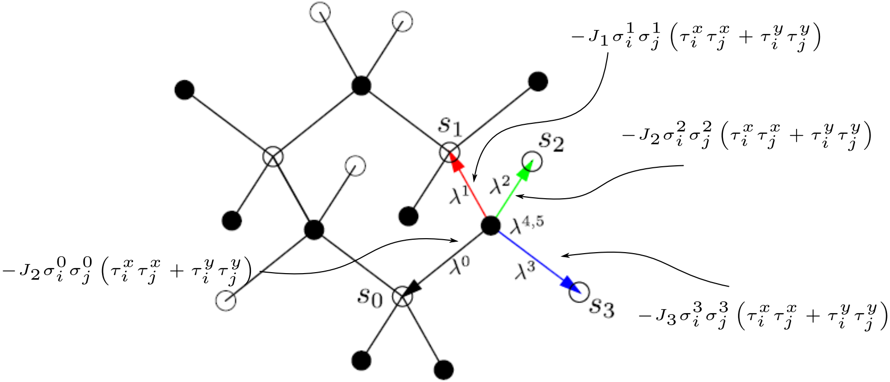
$$H = i\sum_{a=1}^{3} J_a \sum_{i,j} u_{i,j}\lambda_i^4\lambda_j^4$$

$$[H, u_{jk}] = 0 \quad u_{jk}^2 = 0 \Rightarrow u_{jk} = \pm 1$$

Kitaev type model on the diamond lattice

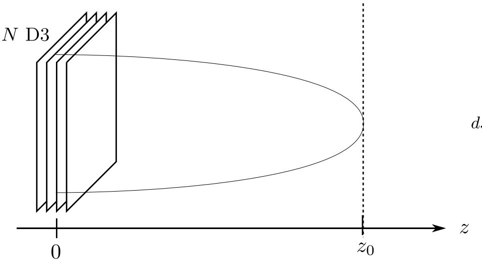
"spin-orbit" Kitaev model ("gamma matrix" Kitaev model) SR (2009)

$$H = -\sum_{\mu=0}^{3} J_\mu \sum_{\mu-\text{links}} \sigma_i^\mu \sigma_j^\mu \left(\tau_i^x \tau_j^x + \tau_i^y \tau_j^y \right)$$



interacting topological phase in symmetry class DIII

AdS/CFT, AdS/CS, AdS/TIS, TSC

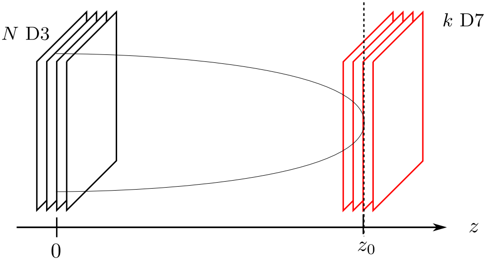


holographic dual of pure YM in (2+1)D

Witten (98)

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + f(z)dy^2 + dx_1^2 + dx_2^2 + f^{-1}(z)dz^2\right)$$

$$f(z) = 1 - (z/z_0)^4$$



holographic dual of interacting topological phase

$$S_{\mathrm{D3}} = \frac{k}{4\pi}\int \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

$$S_{top} \sim \frac{k^2}{2}\log N$$

Fujita, Li, SR, Takayanagi(2009)

- Complete classification of topological phases in fermion systems
  in all dimensions and symmetry classes

bulk-to-boudary approach, K-theory, D-branes, dimensional reduction ... :
all agree

a big open issue: interactions

- do non-interacting topological phases survive interactions ?
- can topological phases arise solely due to interactions ?
- is there "fractional" topological insulators/superconductors ?
- is there a topological classification for bosonic systems (e.g., spin systems)

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Andreas Ludwig (Santa Barbara)
Ashvin Vishwanath (Berkeley)